North Sydney Boys High School — 100 Years 1912 ~ 2012

MATHEMATICS (EXTENSION 1)

2014 HSC Course Assessment Task 2

Wednesday March 12, 2014

General instructions

- Working time – 55 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 5)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED:** .....

Class (please ✔)

- ◯ 12M3A – Mr Zuber
- ◯ 12M3B – Mr Berry
- ◯ 12M3C – Mr Lowe
- ◯ 12M4A – Ms Ziaziaris
- ◯ 12M4B – Mr Lam
- ◯ 12M4C – Mr Ireland

Marker's use only.

QUESTION	1-4	5	6	7	8	9	Total	%
MARKS	$\overline{4}$	$\overline{6}$	$\overline{10}$	$\overline{11}$	$\overline{10}$	$\overline{9}$	$\overline{50}$	

Section I: Objective response

Mark your answers on the multiple choice sheet provided. **Marks**

1. How many real solutions are there to the equation $e^{2x} + e^x - 6 = 0$? **1**

(A) 0

(B) 1

(C) 2

(D) None of the above

2. Which of the following represent the solution(s) to the equation **1**

$$\ln(x-2) + \ln(x-1) = \ln(x+2)$$

(A) $x = 0$, $x = 4$.

(B) $x = 4$ only

(C) $x = 0$ only

(D) No real solutions

3. Which of the following represents the primitive of π^x, excluding the constant of integration? **1**

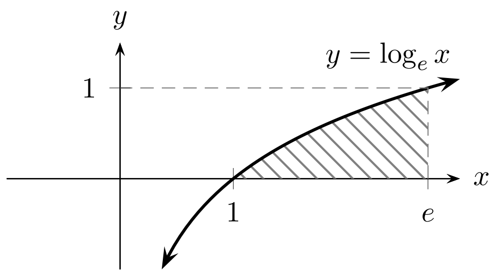
(A) x^π

(B) π^x

(C) $\pi^x \ln \pi$

(D) $\dfrac{\pi^x}{\ln \pi}$

4. Which of the following represents the correct integral to evaluate the volume generated when the area beneath the curve $y = \log_e x$ between $x = 1$ and $x = e$ is rotated about the y axis? **1**



(A) $\pi \displaystyle\int_0^1 e^{2y}\, dy$

(B) $\pi e^2 - \pi \displaystyle\int_0^1 e^{2y}\, dy$

(C) $\pi \displaystyle\int_1^e (\log_e x)^2\, dx$

(D) $\pi e^2 - \pi \displaystyle\int_1^e (\log_e x)^2\, dx$

End of Section I.
Examination continues overleaf.

Question 5 (6 Marks) Commence a NEW page. **Marks**

(a) Find the set of values of x for which the limiting sum exists for this series: **2**

$$1 + \ln x + (\ln x)^2 + (\ln x)^3 + \cdots$$

(b) i. Copy and fill in the table of values for $y = \log_e x - 1$. **1**

x	3	4	5	6	7
y					

ii. By using Simpson's Rule with five function values, find the approximate volume when the curve $y = f(x)$ is rotated about the x axis between $x = 3$ to $x = 7$, correct to 4 decimal places. **3**

Question 6 (10 Marks) Commence a NEW page. **Marks**

(a) Find $\displaystyle\lim_{x\to 0} \frac{\sin 9x}{\tan 4x}$. **2**

(b) i. State the period and amplitude of $y = 2 - \cos \frac{1}{2}x$. **2**

ii. Hence, sketch the graph of $y = 2 - \cos \frac{1}{2}x$ where $-\pi \le x \le \pi$. **2**

(c) i. Sketch the graph of $y = \tan x$ for $-\pi \le x \le \pi$. **2**

ii. Hence on the same set of axes, sketch the graph of $y = \cot x$. **2**

Question 7 (11 Marks) Commence a NEW page. **Marks**

(a) Find the derivative of $y = \log_e \left(\sqrt{\dfrac{1 - x^2}{1 + x^2}} \right)$. **3**

(b) i. Differentiate $\dfrac{2x^2 + 1}{3x^2 - 4}$ with respect to x. **2**

ii. Hence evaluate $\displaystyle\int \frac{x}{(3x^2 - 4)^2}\, dx$. **2**

(c) i. Show that $\dfrac{x^3}{x + 1} = x^2 - x + 1 - \dfrac{1}{x + 1}$. **2**

ii. Hence evaluate $\displaystyle\int \frac{x^3 - 2}{x + 1}\, dx$. **2**

Question 8 (10 Marks) Commence a NEW page. **Marks**

For the curve $y = xe^x + e^x$,

(a) Find the stationary point on the curve and determine its nature. **3**

(b) Find any points of inflexion. **3**

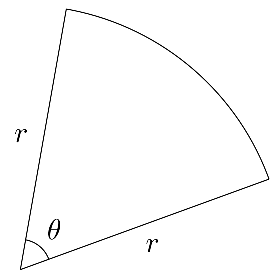
(c) Find where the curve cuts the coordinate axes. **2**

(d) Hence sketch the curve $y = e^x + xe^x$. **2**

Question 9 (9 Marks) Commence a NEW page. **Marks**

(a) Prove that $4 \times 2^n + 3^{3n}$ is divisible by 5 for all integers n, $n \geq 1$. **3**

(b) The diagram shows a sector of a circle with radius r cm. the angle at the centre is θ radians, and the area is $32\,\text{cm}^2$.



i. Find an expression for r in terms of θ. **1**

ii. Show that the perimeter P of the sector is given by **2**

$$P = \frac{8(2+\theta)}{\sqrt{\theta}}$$

iii. Find the minimum perimeter and the value of θ for which this occurs. **3**

End of paper.

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

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1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x, \ x > 0$

Suggested Solutions

Section I

1. (B) **2.** (B) **3.** (D) **4.** (B)

Section II

Question 5 (Lam)

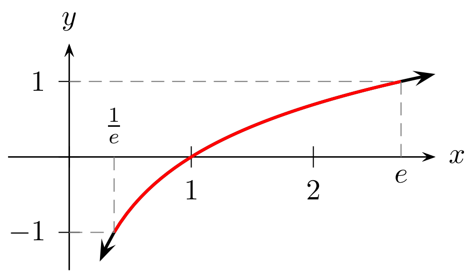
(a) (2 marks)

$$1 + \ln x + (\ln x)^2 + (\ln x)^3 + \cdots$$
$$a = 1 \qquad r = \ln x$$

Limiting sum exists when $|r| < 1$, i.e.

$$|\ln x| < 1$$

From the graph of $y = \ln x$,



the part of $\ln x$ that lies between -1 and 1 occurs when $\frac{1}{e} < x < e$.

(b) i. (1 mark)

- Award full marks for either 3-4 decimal places required for full marks.

x	3	4	5	6	7
y	$\ln 3 - 1$ 0.0986	$\ln 4 - 1$ 0.3863	$\ln 5 - 1$ 0.6094	$\ln 6 - 1$ 0.7918	$\ln 7 - 1$ 0.9459

ii. (3 marks)

$$V = \pi \int_a^b y^2\,dx$$

$$\begin{aligned} V &\approx \pi \times \overbrace{\frac{h}{3}\left((y_0)^2 + 4\left(\sum (y_{\text{odd}})^2\right)\right.}^{\approx \int_a^b y^2\,dx} \\ &\qquad \left. +2\left(\sum (y_{\text{even}})^2\right) + (y_\ell)^2\right) \\ &= \frac{\pi}{3}\left((\ln 3 - 1)^2\right. \\ &\qquad + 4\left((\ln 4 - 1)^2 + (\ln 6 - 1)^2\right) \\ &\qquad \left. +2\,(\ln 5 - 1)^2 + (\ln 7 - 1)^2\right) \\ &= 4.97599\cdots \approx 4.9760 \end{aligned}$$

Question 6 (Ziaziaris)

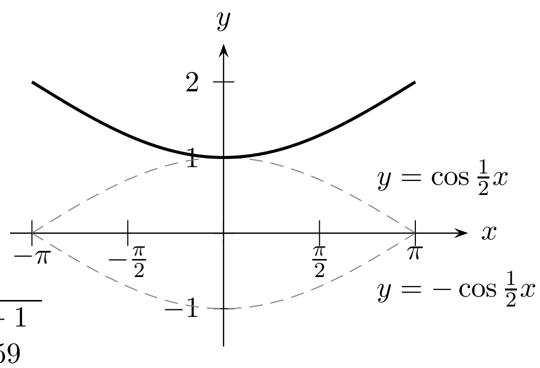
(a) (2 marks)

$$\begin{aligned} \lim_{x\to 0}\frac{\sin 9x}{\tan 4x} &= \lim_{x\to 0}\left(\frac{\sin 9x}{9x} \times \frac{4x}{\tan 4x}\right) \times \frac{9}{4} \\ &= \frac{9}{4} \end{aligned}$$

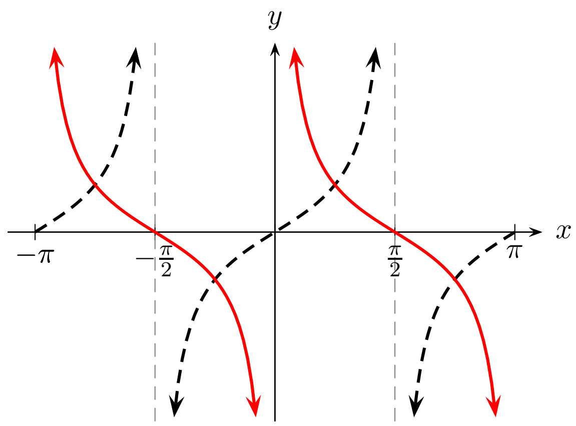
(b) i. (2 marks) $y = 2 - \cos \frac{1}{2}x$

- $T = 4\pi$
- $a = 1$

ii. (2 marks)



(c) (4 marks)



(Dashed line: $y = \tan x$)

Question 7 (Zuber)

(a) (3 marks)

$$\begin{aligned} y &= \ln\sqrt{\frac{1-x^2}{1+x^2}} \\ &= \frac{1}{2}\ln\left(1-x^2\right) - \frac{1}{2}\ln\left(1+x^2\right) \end{aligned}$$

Differentiating,

$$\begin{aligned} \frac{dy}{dx} &= \frac{-\cancel{2}x}{\cancel{2}(1-x^2)} - \frac{\cancel{2}x}{\cancel{2}(1+x^2)} \\ &= -\frac{x}{1-x^2} - \frac{x}{1+x^2} \end{aligned}$$

(b) i. (2 marks)

$$y = \frac{2x^2+1}{3x^2-4}$$

$$\left|\begin{array}{ll} u = 2x^2+1 & v = 3x^2-4 \\ u' = 4x & v' = 6x \end{array}\right.$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{4x\left(3x^2-4\right) - 6x\left(2x^2+1\right)}{\left(3x^2-4\right)^2} \\ &= \frac{\cancel{12x^3} - 16x - \cancel{12x^3} - 6x}{\left(3x^2-4\right)^2} \\ &= -\frac{22x}{\left(3x^2-4\right)^2} \end{aligned}$$

ii. (2 marks)

$$\begin{aligned} \int \frac{x}{\left(3x^2-4\right)^2}\,dx &= -\frac{1}{22}\int \frac{-22x}{\left(3x^2-4\right)^2}\,dx \\ &= -\frac{1}{22}\left(\frac{2x^2+1}{3x^2-4}\right) + C \end{aligned}$$

(c) i. (2 marks)

$$\begin{aligned} \frac{x^3}{x+1} &= \frac{x^3+1 \quad -1}{x+1} \\ &= \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{x+1}} - \frac{1}{x+1} \\ &= x^2 - x + 1 - \frac{1}{x+1} \end{aligned}$$

ii. (2 marks)

$$\begin{aligned} \int \frac{x^3-2}{x+1}\,dx &= \int \left(\frac{x^3}{x+1} - \frac{2}{x+1}\right)\,dx \\ &= \int \left(x^2 - x + 1 - \frac{3}{x+1}\right)\,dx \\ &= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 3\ln(x+1) + C \end{aligned}$$

Question 8 (Ireland)

(a) (3 marks)

$$y = xe^x + e^x = e^x(x+1)$$

$$\begin{array}{ll} u = e^x & v = x+1 \\ u' = e^x & v' = 1 \end{array}$$

$$\begin{aligned} \frac{dy}{dx} &= e^x + e^x(x+1) \\ &= e^x(x+2) \end{aligned}$$

Stationary points occur when $\frac{dy}{dx} = 0$ $(e^x > 0)$

$$\therefore x = -2 \qquad y = -e^{-2}$$

Test for type of statioary point. Find second derivative:

$$\begin{array}{ll} u = e^x & v = x+2 \\ u' = e^x & v' = 1 \end{array}$$

$$\frac{d^2y}{dx^2} = e^x + e^x(x+2) = e^x(x+3)$$

When $x = -2$,

$$\frac{d^2y}{dx^2} = e^{-2}(-2+3) > 0$$

Hence $\left(-2, -e^{-2}\right)$ is a local minimum.

(b) (3 marks) Point(s) of inflexion occur when $\frac{d^2y}{dx^2} = 0$:

$$\therefore x = -3 \qquad y = e^{-3}(-3+1) = -2e^{-3}$$

Testing for sign change:

- $x = -4$, $\dfrac{d^2y}{dx^2} = e^{-4}(-4+1) < 0$
- $x = -2$, $\dfrac{d^2y}{dx^2} > 0$

Hence $\left(-3, -2e^{-3}\right)$ is a point of inflexion.

(c) (2 marks)

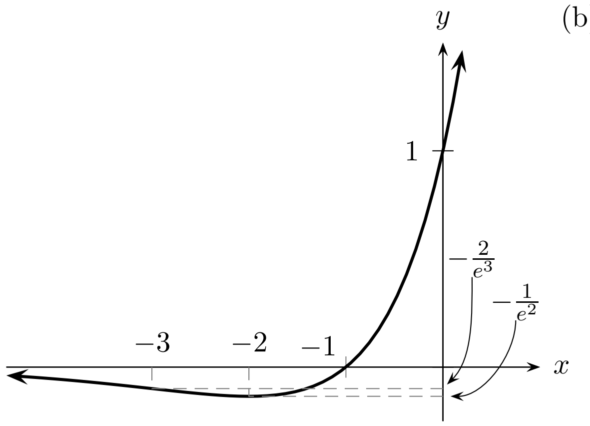
- $x = 0$

$$y = e^0(0+1) = 1$$

- $y = 0$,

$$e^x(x+1) = 0$$
$$\therefore x = -1$$

(d) (2 marks)



Question 9 (Lam)

(a) (3 marks) Let $P(n)$ be the proposition

$$P(n) : 4 \times 2^n + 3^{3n} = 5M$$

where $M \in \mathbb{Z}^+$.

- Base case: $P(1)$:

$$4 \times 2 + 3^3 = 8 + 27 = 35$$

Hence $P(1)$ is true.

- Inductive step: assume $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e.

$$P(k) : 4 \times 2^k + 3^{3k} = 5N$$

$(N \in \mathbb{Z}^+)$

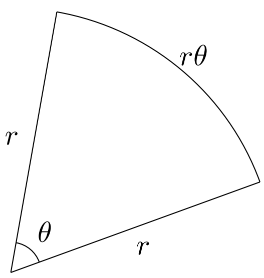
Examine $P(k+1)$:

$$\begin{aligned} & 4 \times 2^{k+1} + 3^{3(k+1)} \\ &= \left(4 \times 2^k\right) \times 2^1 + 3^{3k} \times 3^3 \\ &= \left(5N - 3^{3k}\right) \times 2 + 3^{3k} \times 3^3 \\ &= 5 \times 2N - 2 \times 3^{3k} + 27 \times 3^{3k} \\ &= 5 \times 2N + 25 \times 3^{3k} \\ &= 5\left(2N + 5 \times 3^{3k}\right) = 5P \end{aligned}$$

where $P \in \mathbb{Z}^+$

$\therefore P(k+1)$ is also true, and $P(n)$ is true by induction.

(b) i. (1 mark)



$$A = 32 = \frac{1}{2}r^2\theta$$
$$\therefore r^2 = \frac{64}{\theta}$$
$$r = \frac{8}{\sqrt{\theta}}$$

ii. (2 marks)

$$P = 2r + r\theta = r(2+\theta)$$
$$= \left(\frac{8}{\sqrt{\theta}}\right)(2+\theta)$$

iii. (3 marks)

$$P = 8\theta^{-\frac{1}{2}}(2+\theta)$$

Differentiating,

$$\left| \begin{array}{ll} u = 8\theta^{-\frac{1}{2}} & v = 2+\theta \\ u' = -\frac{1}{2} \times 8\theta^{-\frac{3}{2}} & v' = 1 \\ \quad = -4\theta^{-\frac{3}{2}} & \end{array} \right.$$

$$\begin{aligned} \frac{dP}{d\theta} &= 8\theta^{-\frac{1}{2}} - 4\theta^{-\frac{3}{2}}(2+\theta) \\ &= 8\theta^{-\frac{1}{2}} - 8\theta^{-\frac{3}{2}} - 4\theta^{-\frac{1}{2}} \\ &= 4\theta^{-\frac{1}{2}} - 8\theta^{-\frac{3}{2}} \\ &= 4\theta^{-\frac{1}{2}}\left(1 - 2\theta^{-1}\right) \\ &= \frac{4}{\sqrt{\theta}}\left(1 - \frac{2}{\theta}\right) \end{aligned}$$

Stationary points occur when $\frac{dP}{d\theta} = 0$:

$$\begin{aligned} \frac{4}{\sqrt{\theta}}\left(1 - \frac{2}{\theta}\right) &= 0 \\ \left(1 - \frac{2}{\theta}\right) &= 0 \\ \therefore \frac{2}{\theta} &= 1 \\ \theta &= 2 \end{aligned}$$

Check second derivative:

$$\begin{aligned} \frac{d^2P}{d\theta^2} &= -\frac{1}{2} \times 4\theta^{-\frac{3}{2}} - \left(-\frac{3}{2}\right) \times 8\theta^{-\frac{5}{2}} \\ &= -2\theta^{-\frac{3}{2}} + 12\theta^{-\frac{5}{2}}\Big|_{\theta=2} \\ &= \frac{-2}{\sqrt{2^3}} + \frac{12}{\sqrt{2^5}} \approx 1.414 \\ &> 0 \end{aligned}$$

Hence $\theta = 2$ is a local minimum. P is minimised when $\theta = 2$.

$$P = \frac{8}{\sqrt{2}}(2+2) = \frac{32}{\sqrt{2}}$$